

Chip Firing

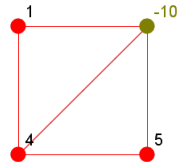
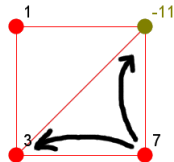
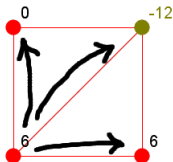
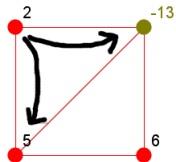
Alex St Laurent

work with Caroline Klivans
Division of Applied Mathematics
Brown University

ICERM Undergraduate Mathematics Research
Presentations, 2012

The Graphical Case

- Each vertex holds an integer value (number of chips).
- A vertex can **fire** if it has enough chips to share.
- A **bank** vertex can fire iff none of the others are able.
- Computationally, behavior defined by Laplacian matrix.



The Graphical Case

- A configuration of chips is . . .
 - **stable** if only the bank can fire.
 - **recurrent** if there is a sequence of firings that returns to the configuration.
 - **critical** if it is both stable and recurrent.
- Every configuration can be fired to reach a critical configuration.
- Critical configurations form a group structure.
- Order of group equals number of spanning trees.
- dollar game, abelian sandpile model, BTW sandpile model

The General Case

- Graphs are *one-dimensional*.
- In general, chips on $(i - 1)$ -dimensional structures, firing across i -dimensional structures.
- The bank is a tree that spans the $(i - 2)$ -dimensional structures.

- Behavior still defined by a Laplacian.
- Known: the **critical group** exists.
- Unknown: how to define critical configurations in general (i.e. the ideas of stability and recurrence).

The Program

- To aid experimentation, time was devoted to building an application.
- Written in Java, makes use of Processing.org libraries.
- Visualize the two-dimensional case.



Discrete Flow

BUILD ▾

New

Load

Save



Labels



Degree



Config (V)



Config (E)

7



Vertex Size

5.00



Arrow Size

15



Grid Size

TREE ▸

CONFIG ▸

RUN ▸

CTRL+A - select all
CTRL+N - select none
CTRL+Z - undo
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CTRL+S - save
L - toggle edge between TWO
selected vertices
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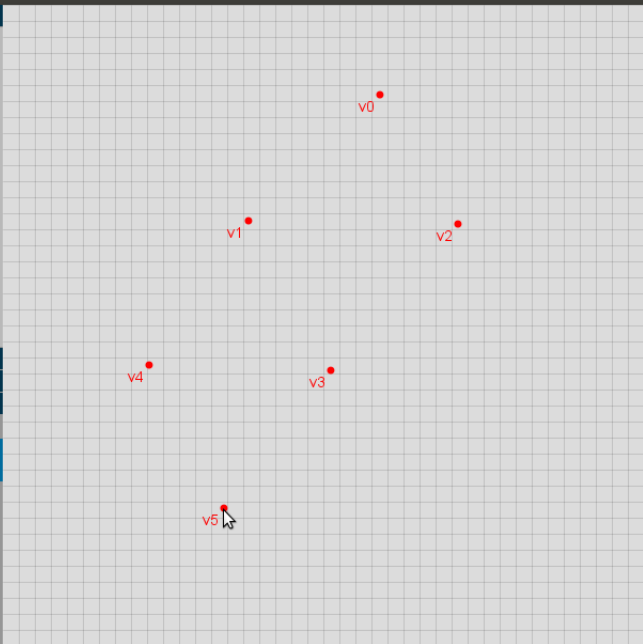
Grid Size

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v0

v1

v2

v3

v4

v5



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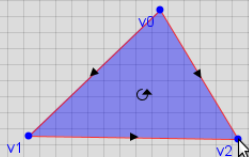
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v4

v3

v5



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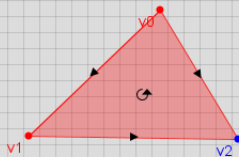
Grid Size

TREE ▸

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RUN ▸

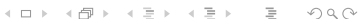
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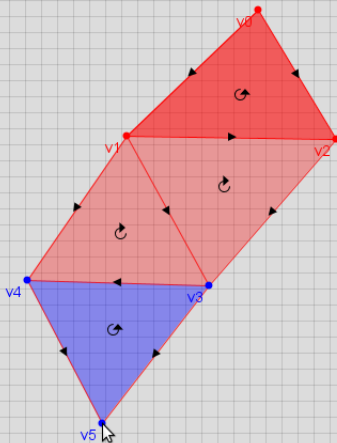
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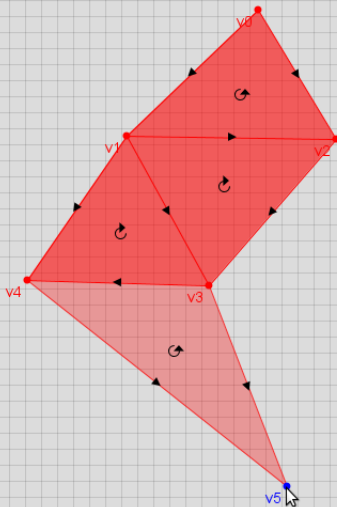
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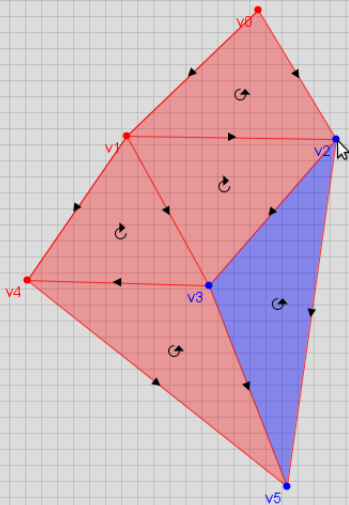
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Discrete Flow

BUILD

TREE

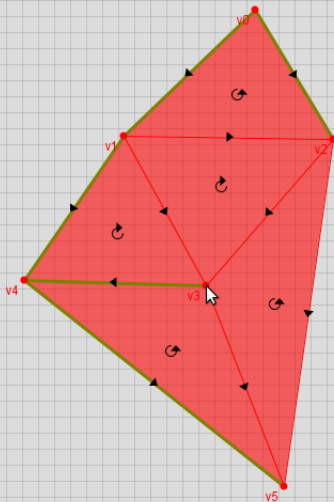


1D 2D

CONFIG

RUN

CLICK to select
C - clear current tree
L - toggle tree-ness of edge
between TWO selected vertices
CTRL+N - select none





Discrete Flow

BUILD ^

TREE ^

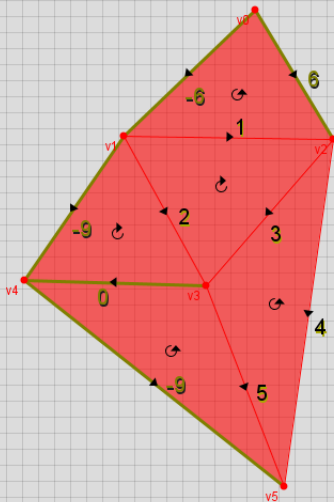
CONFIG v



Value(s)

RUN ^

CTRL+A - select all
CTRL+N - select none
CTRL+Z - undo



Discrete Flow

BUILD ^

TREE ^

CONFIG ^

RUN v

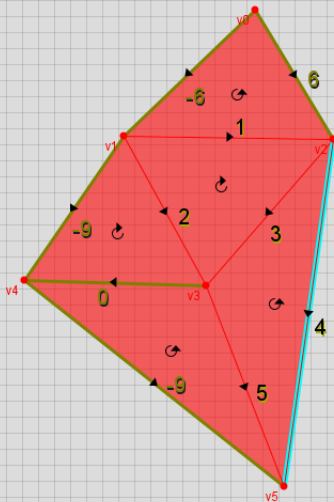
FIRE

TREE

Selector

MAP

CTRL+A - focus all
CTRL+N - focus none
CTRL+Z - undo
TAB - fire selection
SHIFT+TAB - water selection
CTRL+M - focus "nice" edges
CTRL+P - focus only positive edges
L - toggle focus of edge between TWO selected vertices



Discrete Flow

BUILD ^

TREE ^

CONFIG ^

RUN v

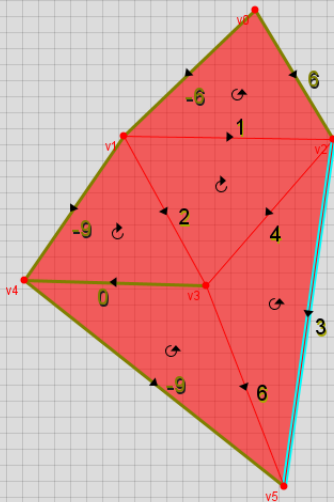
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Discrete Flow

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TREE

CONFIG

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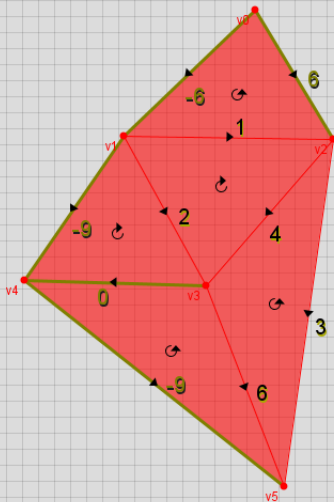
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Discrete Flow

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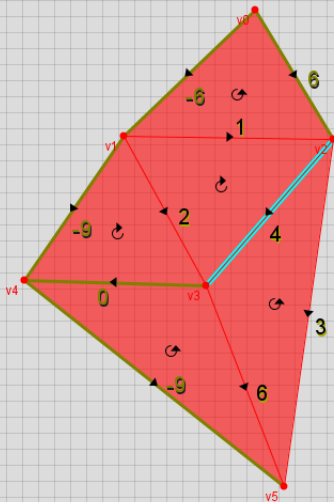
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Discrete Flow

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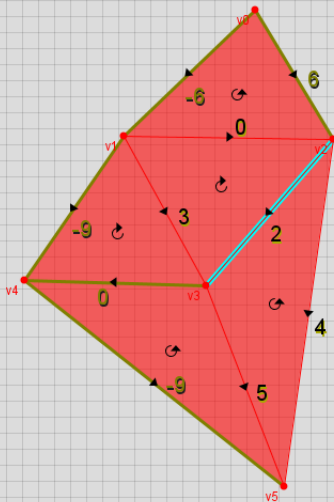
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Discrete Flow

BUILD ^

TREE ^

CONFIG ^

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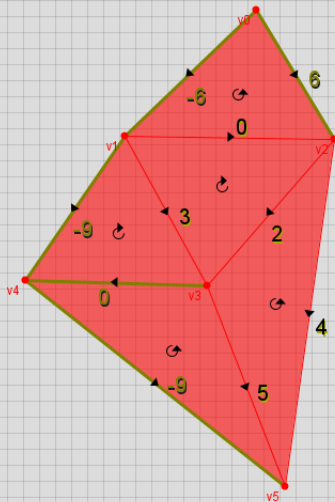
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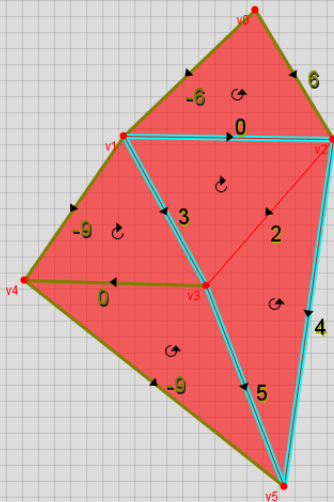
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3



Discrete Flow

BUILD ^

TREE ^

CONFIG ^

RUN v

FIRE

TREE

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