Negative Snell's Law

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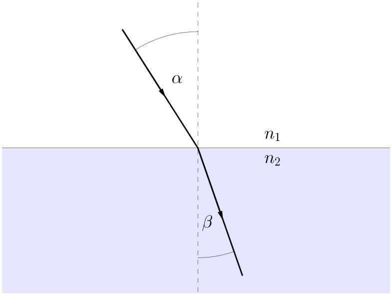
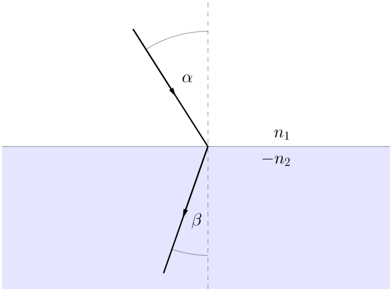
Snell's Law



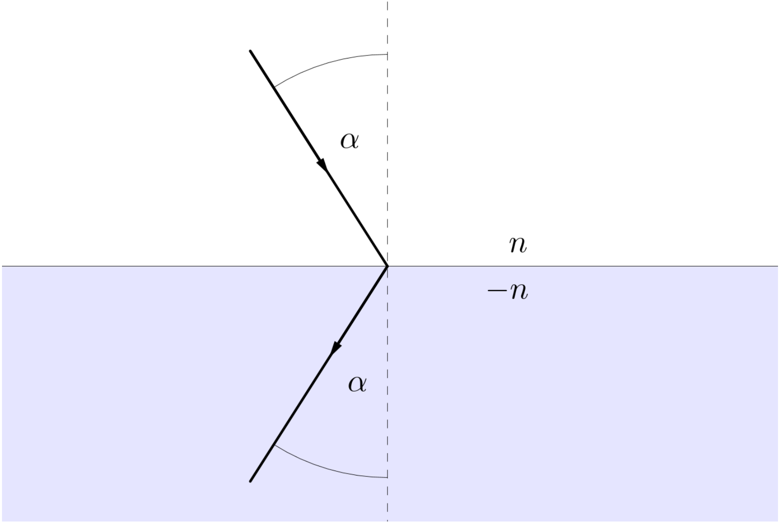
Positive to positive



Positive to negative

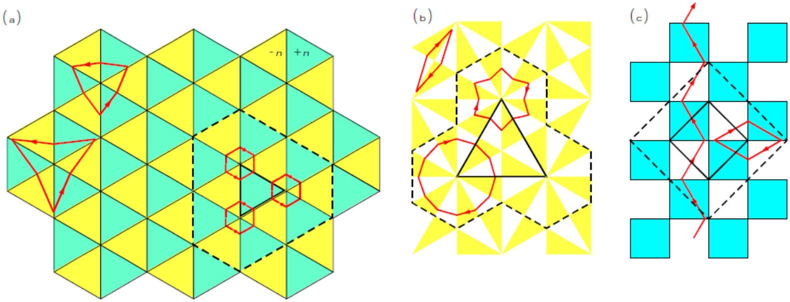
$$\frac{\sin \alpha}{\sin \beta} = \frac{n_2}{n_1}$$

Equal and Opposite



The angle is reflected across the boundary

As a Dynamical System



(a) Equilateral triangle tiling, (b) 30°-60°-90° triangle tiling, (c) square tiling. Images from Mascarenhas, Fluegel.

Questions

Given a tiling, . . .

- when do *periodic* orbits occur?
- when do *drift periodic* orbits occur?
- what is the nature of the *stability* of these orbits?

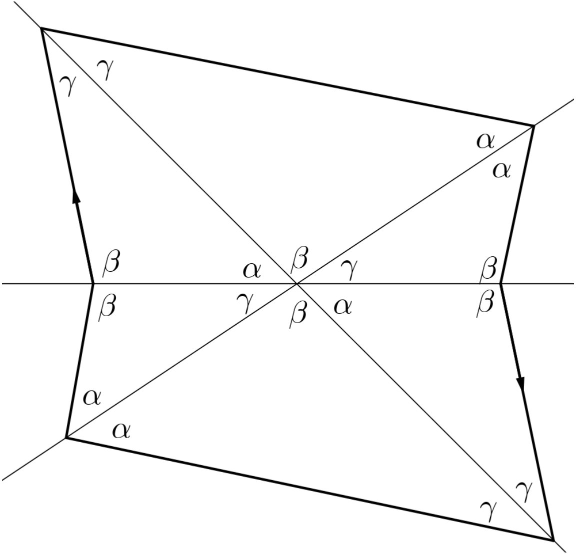
Some Previous Results

Theorem (Mascarenhas, Fluegel)
In the equilateral triangle tiling, and the in the square tiling, every trajectory is either periodic or drift periodic.

Theorem (Engelman, Kimball)

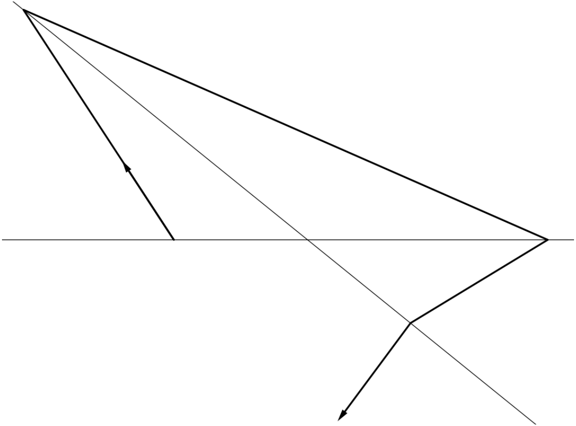
- *In tilings with* **three lines** *meeting at a point, there is* **always** *a periodic orbit.*
- *There are* **no** *periodic orbits about the intersection of* **two non-perpendicular lines**.

Some Previous Results



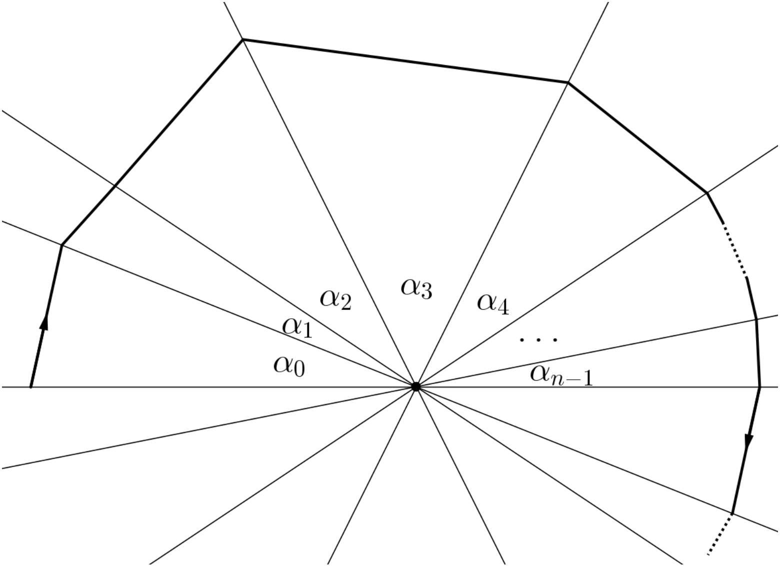
A periodic orbit about the intersection of three lines

Some Previous Results



A periodic orbit is impossible here

Extending These Results: n Coincident Lines

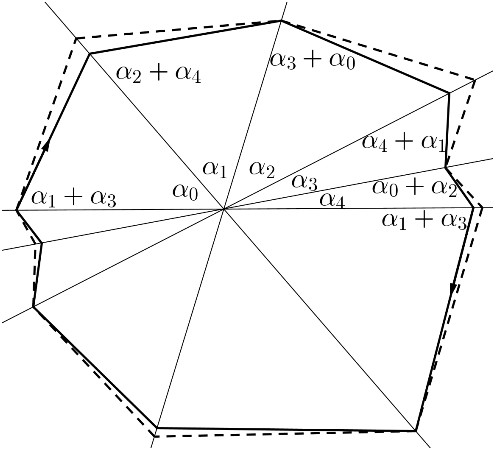


For *n* Odd

Theorem

In tilings with an **odd number of lines** *meeting at a point, there is always an orbit which is* **periodic** *and* **stable**

For n Odd



A stable periodic orbit about five lines

For n Even

Theorem

In tilings with an **even number of lines** *meeting at a point, there is only a periodic orbit if:*

$$(\alpha_{n-2} - \alpha_{n-1}) + \cdots + (\alpha_2 - \alpha_3) + (\alpha_0 - \alpha_1) = 0$$

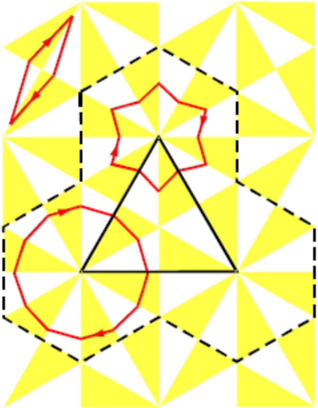
If there is a periodic orbit, it will be **stable***.*

Corollary

A periodic orbit around two lines is only possible if they are **perpendicular***.*

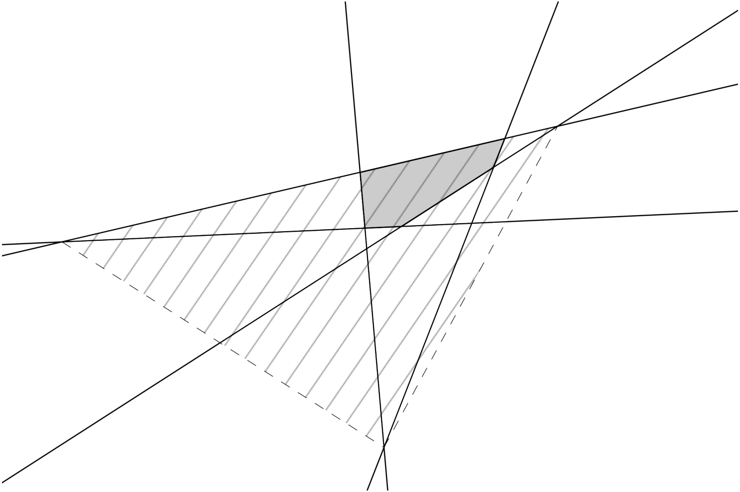
- $(\alpha_0 - \alpha_1) = 0 \Rightarrow \alpha_0 = \alpha_1 = \dfrac{\pi}{2}$

For n Even



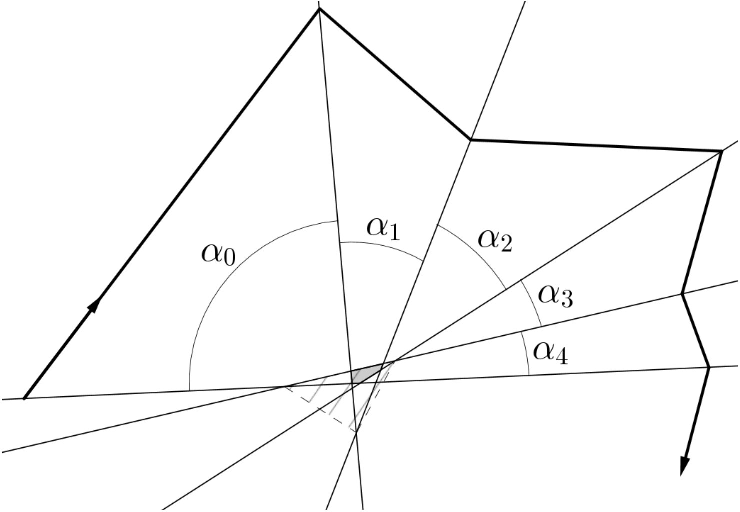
Periodic orbits about four lines (upper left) and six lines. Image from Mascarenhas, Fluegel.

Extending Further: n Lines at an Polygon



The edges of the shaded pentagon extended into lines. The convex hull of the intersection points is the hatched region.

n Lines at a Polygon



A trajectory staying outside the convex hull.

For *n* Odd

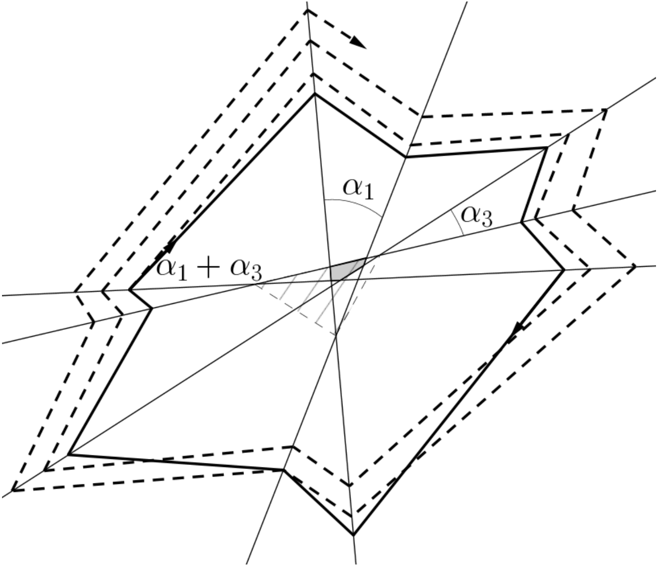
Theorem

When there are an **odd number of non-parallel lines** *meeting at a polygon, there is always an orbit which is* **periodic** *and* **unstable***.*

Corollary

When there are an **odd number of lines** *meeting at a* **regular polygon***, there is always an orbit which is* **periodic** *and* **unstable***.*

For *n* Odd



An unstable periodic orbit

For n Even

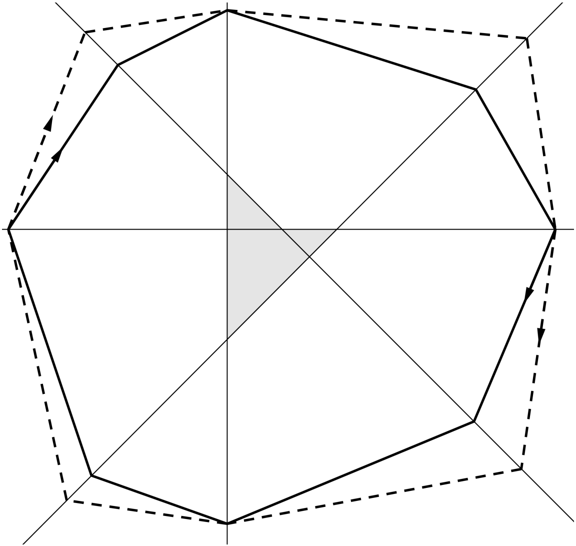
Theorem

If there are an **even number of non-parallel lines** *meeting at a polygon, there is only a periodic orbit if:*

$$(\alpha_{n-2} - \alpha_{n-1}) + \cdots + (\alpha_2 - \alpha_3) + (\alpha_0 - \alpha_1) = 0$$

If there is a periodic orbit, it will be **stable**.

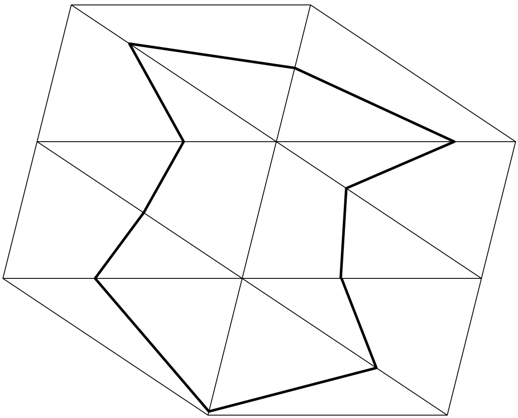
For n Even



A stable periodic orbit

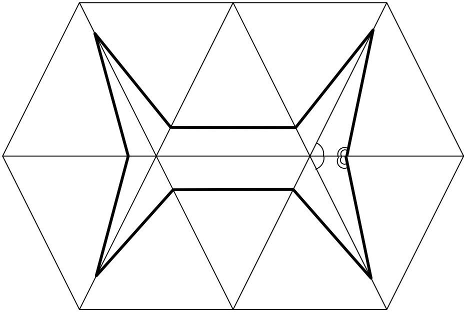
Scalene Triangle Tilings

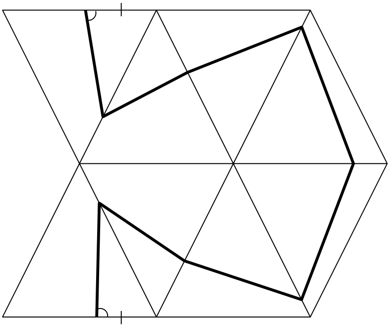
Every triangle tiling has a 10-periodic orbit (circles two vertices) except for isosceles triangle tilings with equal angles less than or equal to $\frac{\pi}{3}$.



Isosceles Triangle Tilings

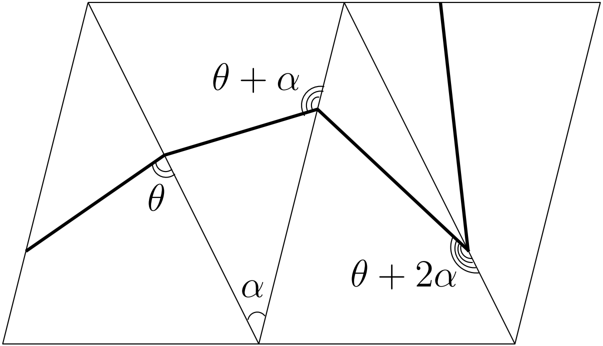
Orbits in an isosceles triangle tiling are either periodic or drift-periodic, and the period of an orbit in a given tiling is bounded by $2n + 4$, for $n \in \mathbb{N}$ such that $\frac{\pi}{\alpha} - 1 \leq n < \frac{\pi}{\alpha}$.





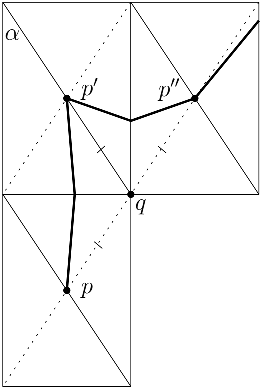
Angle Adding Lemma

Let α be an angle of the tiling triangle. If an orbit only hits the legs forming angle α, then each time the orbit meets a leg, the angle it makes with the leg on the side away from α will increase by α.



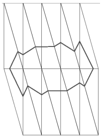
Right Triangle Tilings

Orbits that bisect a hypotenuse in a right triangle tiling will bisect every hypotenuse they meet, and therefore will escape to infinity.

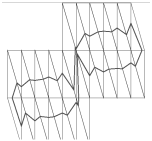


These orbits are stable under perturbation from the midpoint if $\alpha = \frac{p\pi}{q}$ for some $\frac{p}{q} \in \mathbb{Q}$.

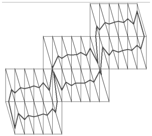
Observations



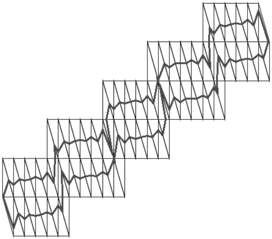
The largest periodic orbit contained in a single row.



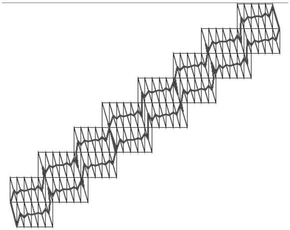
The two-row periodic orbit.



The three-row periodic orbit.



The five-row periodic orbit.

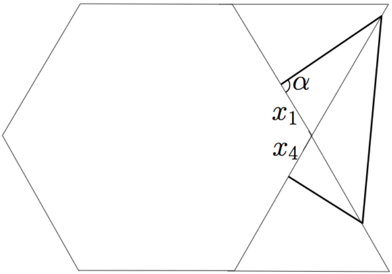


The eight-row periodic orbit.

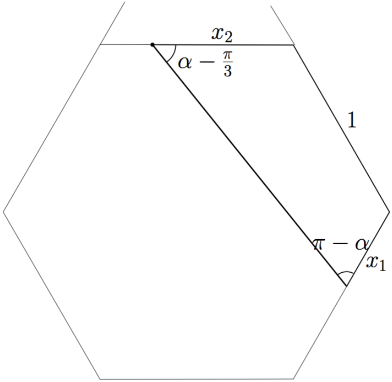
Further Questions on Triangle Tilings

- In multi-row periodic orbits, why do the orbits always stay in a row for the maximum or near-maximum number of edges?
- How many rows can a multi-row periodic orbit traverse? Is there a pattern?
- What is the pattern for the number of edges that a multi-row periodic orbit crosses in each row?

Common Trajectory Paths in the Trihexagonal Tiling

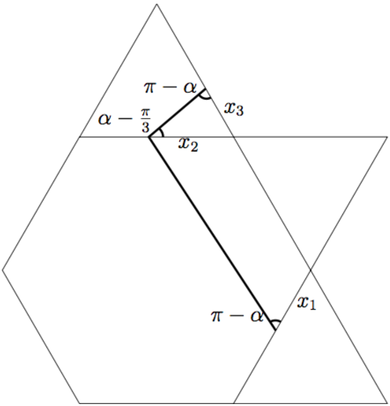


$x_1 = x_4$

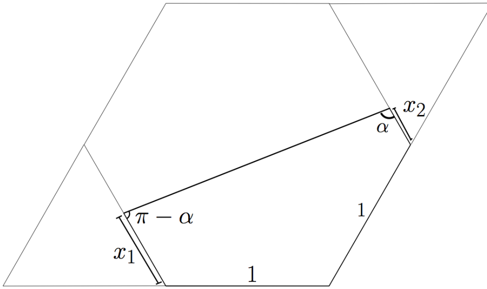


$$x_2 = \frac{\sin\alpha(1+2x_2)+\sqrt{3}\cos\alpha}{\sin(\alpha-\frac{2\pi}{3})}.$$

Common Trajectory Paths in the Trihexagonal Tiling



$$x_2 = \frac{1}{2} + x_1 + \frac{\sqrt{3}}{2} \cot \alpha$$



$$x_2 = x_1 + \sqrt{3} \cot \alpha$$

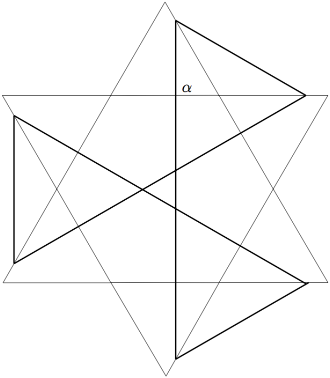
Periodic Paths

Theorem

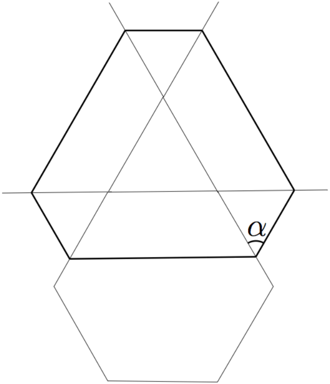
There exists a 12-periodic orbit with initial angle $\alpha = \frac{\pi}{2}$

Theorem

There exists a 6-periodic orbit with the initial angle $\alpha = \frac{2\pi}{3}$



The 12-periodic path.

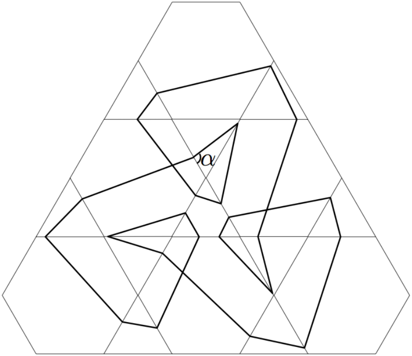


The 6-periodic path

Periodic Paths

Theorem

There exists a 24-period periodic orbit with an initial angle of $\alpha = \pi - \tan^{-1}(2\sqrt{3})$.

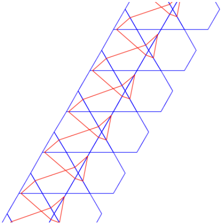


A 24-periodic path made of 3 turning trajectories

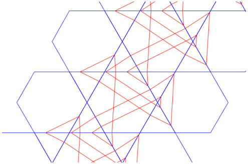
Drift-Periodic Paths

Theorem

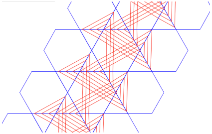
Given an initial trajectory forming an angle $\alpha = \pi - \tan^{-1}((6n-3)\sqrt{3})$ *with* $n \in \mathbb{N}$ *to a side of the hexagon, there exists a drift periodic orbit where n equals the maximum number of intersections to a side of the hexagon.*



n = 1.



n = 2

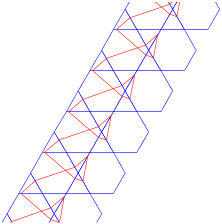


n=4

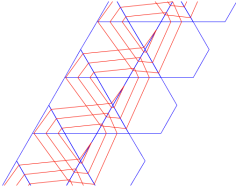
Drift-Periodic Paths

Theorem

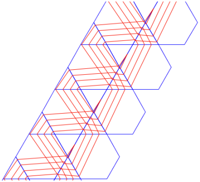
Given an initial angle $\alpha = \pi - \tan^{-1}(\frac{3n\sqrt{3}}{3n-2})$*, then there exists a drift-periodic path where n equals the maximum number of intersections to a side of the hexagon.*



n = 1

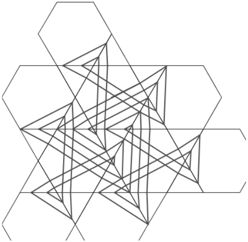


n = 3

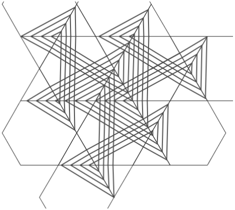


n=4

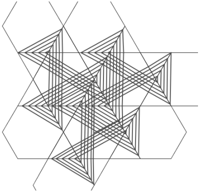
Periodic Families



Period 78

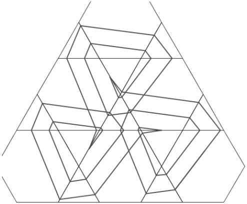


Period 138

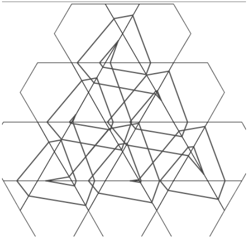


Period 174

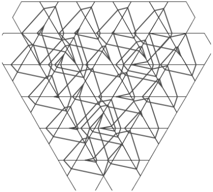
Periodic Families



Period 42



Period 66



Period 192

Further Questions on the Trihexagonal Tiling

- Can the existence of the previous periodic families be proven?
- How many families of periodic orbits exist?
- Does every family of periodic orbits also have a family of drift-periodic orbits?
- Are there any paths that are stable under angle shifts?

Acknowledgments

Diana Davis and Chaim Goodman-Strauss for guidance and support. Pat Hooper for sharing his the source code of his Java program from last year.

References

(1) A. Mascarenhas, B. Fluegel: *Antisymmetry and the breakdown of Bloch's theorem for light*, preprint.

(2) K. Engelman, A. Kimball: *Negative Snell's Law*, preprint, 2012.